

# ON CODIMENSION TWO SUBVARIETIES IN HYPERSURFACES

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**ABSTRACT.** We show that for a smooth hypersurface  $X \subset \mathbb{P}^n$  of degree at least 2, there exist arithmetically Cohen-Macaulay (ACM) codimension two subvarieties  $Y \subset X$  which are not an intersection  $X \cap S$  for a codimension two subvariety  $S \subset \mathbb{P}^n$ . We also show there exist  $Y \subset X$  as above for which the normal bundle sequence for the inclusion  $Y \subset X \subset \mathbb{P}^n$  does not split.

*Dedicated to Spencer Bloch*

## 1. INTRODUCTION

In this note, we revisit some questions of Griffiths and Harris from 1985 [GH]:

**Questions** (Griffiths and Harris). *Let  $X \subset \mathbb{P}^4$  be a general hypersurface of degree  $d \geq 6$  and  $C \subset X$  be a curve.*

- (1) *Is the degree of  $C$  a multiple of  $d$ ?*
- (2) *Is  $C = X \cap S$  for some surface  $S \subset \mathbb{P}^4$ ?*

The motivation for these questions comes from trying to extend the Noether-Lefschetz theorem for surfaces to threefolds. Recall that the Noether-Lefschetz theorem states that if  $X$  is a very general surface of degree  $d \geq 4$  in  $\mathbb{P}^3$ , then  $\text{Pic}(X) = \mathbb{Z}$ , and hence every curve  $C$  on  $X$  is the complete intersection of  $X$  and another surface  $S$ .

C. Voisin very soon [Vo] proved that the second question had a negative answer by constructing counter-examples on any smooth hypersurface of degree at least 2. She also considered a third question:

**Question.** *With the same terminology and when  $C$  is smooth:*

- (3) *Does the exact sequence of normal bundles associated to the inclusions  $C \subset X \subset \mathbb{P}^4$ :*

$$0 \rightarrow N_{C/X} \rightarrow N_{C/\mathbb{P}^4} \rightarrow \mathcal{O}_C(d) \rightarrow 0$$

*split?*

Her counter-examples provided a negative answer to this question as well. The first question, the Degree Conjecture of Griffiths-Harris, is still open. Strong evidence for this conjecture was provided by some elementary but ingenious examples of Kollár ([BCC], Trento examples). In particular he shows that if  $\gcd(d, 6) = 1$  and  $d \geq 4$  and  $X$  is a very general hypersurface of degree  $d^2$  in  $\mathbb{P}^4$ , then every curve on  $X$  has degree a multiple of  $d$ . In the same vein, Van Geemen shows that if  $d > 1$  is an odd number and  $X$  is a very general hypersurface of degree  $54d$ , then every curve on  $X$  has degree a multiple of  $3d$ .

The main result of this note is the existence of a large class of counterexamples which subsumes Voisin's counterexamples and places them in the context of arithmetically Cohen-Macaulay

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(ACM) vector bundles on  $X$ . It is well known that ACM bundles which are not sums of line bundles can be found on any hypersurface of degree at least 2 [BGS], and for such a bundle, say of rank  $r$ , on  $X$ , ACM subvarieties of codimension two can be created on  $X$  by considering the dependency locus of  $r - 1$  general sections. These subvarieties fail to satisfy Questions 2 and 3. We will be working on hypersurfaces in  $\mathbb{P}^n$  for any  $n \geq 4$  and our constructions of ACM subvarieties may not give smooth ones. Hence in Question 3, we will consider the splitting of the conormal sheaf sequence instead.

## 2. MAIN RESULTS

Let  $X \subset \mathbb{P}^n$  be a smooth hypersurface of degree  $d \geq 2$  and let  $Y \subset X$  be a codimension 2 subscheme. Recall that  $Y$  is said to be an arithmetically Cohen-Macaulay (ACM) subscheme of  $X$  if  $H^i(X, I_{Y/X}(\nu)) = 0$  for  $0 < i \leq \dim Y$  and for any  $\nu \in \mathbb{Z}$ . Similarly, a vector bundle  $E$  on  $X$  is said to be ACM if  $H^i(X, E(\nu)) = 0$  for  $i \neq 0, \dim X$  and for any  $\nu \in \mathbb{Z}$ .

Given a coherent sheaf  $\mathcal{F}$  on  $X$ , let  $s_i \in H^0(\mathcal{F}(m_i))$  for  $1 \leq i \leq k$  be generators for the  $\oplus_{\nu \in \mathbb{Z}} H^0(\mathcal{O}_X(\nu))$ -graded module  $\oplus_{\nu \in \mathbb{Z}} H^0(\mathcal{F}(\nu))$ . These sections give a surjection of sheaves  $\oplus_{i=1}^k \mathcal{O}_X(-m_i) \twoheadrightarrow \mathcal{F}$  which induces a surjection of global section  $\oplus_{i=1}^k H^0(\mathcal{O}_X(\nu - m_i)) \twoheadrightarrow H^0(\mathcal{F}(\nu))$  for any  $\nu \in \mathbb{Z}$ .

Applying this to the ideal sheaf  $I_{Y/X}$  of an ACM subscheme of codimension 2 in  $X$ , we obtain the short exact sequence

$$0 \rightarrow G \rightarrow \oplus_{i=1}^k \mathcal{O}_X(-m_i) \rightarrow I_{Y/X} \rightarrow 0,$$

where  $G$  is some ACM sheaf on  $X$  of rank  $k - 1$ . Since  $Y$  is ACM as a subscheme of  $X$ , it is also ACM as a subscheme of  $\mathbb{P}^n$ . In particular,  $Y$  is locally Cohen-Macaulay. Hence  $G$  is a vector bundle by the Auslander-Buchsbaum Theorem (see [Mat] page 155). We will loosely say that  $G$  is associated to  $Y$ .

Conversely, the following Bertini type theorem which goes back to arguments of Kleiman in [Kl] (see also [Ban]) shows that given an ACM bundle  $G$  on  $X$ , we can use  $G$  to construct ACM subvarieties  $Y$  of codimension 2 in  $X$ :

**Proposition 1.** (Kleiman). *Given a bundle  $G$  of rank  $k - 1$  on  $X$ , a general map  $G \rightarrow \oplus_{i=1}^k \mathcal{O}_X(m_i)$  for sufficiently large  $m_i$  will determine the ideal sheaf (up to twist) of a subvariety  $Y$  of codimension 2 in  $X$  with a resolution of sheaves:*

$$0 \rightarrow G \rightarrow \oplus_{i=1}^k \mathcal{O}_X(m_i) \rightarrow I_{Y/X}(m) \rightarrow 0.$$

Since the conclusion of Question 2 implies that of Question 3, we will look at just Question 3, in the conormal sheaf version.

Let  $X$  be a hypersurface of degree  $d$  in  $\mathbb{P}^n$  defined by the equation  $f = 0$ . Let  $X_2$  be the thickening of  $X$  defined by  $f^2 = 0$  in  $\mathbb{P}^n$ . Given a subvariety  $Y$  of codimension 2 in  $X$ , let  $I_{Y/\mathbb{P}}$  (resp.  $I_{Y/X}$ ) denote the ideal sheaf of  $Y \subset \mathbb{P}^n$  (resp.  $Y \subset X$ ). The conormal sheaf sequence is

$$(1) \quad 0 \rightarrow \mathcal{O}_Y(-d) \rightarrow I_{Y/\mathbb{P}}/I_{Y/\mathbb{P}}^2 \rightarrow I_{Y/X}/I_{Y/X}^2 \rightarrow 0.$$

**Lemma 1.** *For the inclusion  $Y \subset X \subset \mathbb{P}^n$ , if the sequence of conormal sheaves (1) splits, then there exists a subscheme  $Y_2 \subset X_2$  containing  $Y$  such that*

$$I_{Y_2/X_2}(-d) \xrightarrow{f} I_{Y_2/X_2} \rightarrow I_{Y/X} \rightarrow 0$$

*is exact. Furthermore,  $fI_{Y_2/X_2}(-d) = I_{Y/X}(-d)$ .*

*Proof.* Suppose sequence (1) splits: then we have a surjection

$$I_{Y/\mathbb{P}} \twoheadrightarrow I_{Y/\mathbb{P}}/I_{Y/\mathbb{P}}^2 \twoheadrightarrow \mathcal{O}_Y(-d)$$

where the first map is the natural quotient map and the second is the splitting map for the sequence. The kernel of this composition defines a scheme  $Y_2$  in  $\mathbb{P}^n$ . Since this kernel  $I_{Y_2/\mathbb{P}}$  contains  $I_{Y/\mathbb{P}}^2$  and hence  $f^2$ , it is clear that  $Y \subset Y_2 \subset X_2$ .

The splitting of (1) also means that  $f \in I_{Y/\mathbb{P}}(d)$  maps to  $1 \in \mathcal{O}_Y$ . We get the commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ 0 & \rightarrow & I_{Y_2/\mathbb{P}} & \rightarrow & I_{Y/\mathbb{P}} & \rightarrow & \mathcal{O}_Y(-d) \rightarrow 0 \\ & & \uparrow f^2 & & \uparrow f & & \uparrow \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{P}}(-2d) & \xrightarrow{f} & \mathcal{O}_{\mathbb{P}}(-d) & \rightarrow & \mathcal{O}_X(-d) \rightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

This induces

$$0 \rightarrow I_{Y/X}(-d) \rightarrow I_{Y_2/X_2} \rightarrow I_{Y/X} \rightarrow 0.$$

In particular, note that  $I_{Y/X}(-d)$  is the image of the multiplication map  $f : I_{Y_2/X_2}(-d) \rightarrow I_{Y_2/X_2}$ .  $\square$

Now assume that  $Y$  is an ACM subvariety on  $X$  of codimension 2. The ideal sheaf of  $Y$  in  $X$  has a resolution

$$0 \rightarrow G \rightarrow \oplus_{i=1}^k \mathcal{O}_X(-m_i) \rightarrow I_{Y/X} \rightarrow 0,$$

for some ACM bundle  $G$  on  $X$  associated to  $Y$ .

**Lemma 2.** *Suppose the conditions of the previous lemma hold, and in addition  $Y$  is an ACM subvariety. Then there is an extension of the ACM bundle  $G$  (associated to  $Y$ ) on  $X$  to a bundle  $\mathcal{G}$  on  $X_2$ . ie. there is a vector bundle  $\mathcal{G}$  on  $X_2$  such that the multiplication map  $f : \mathcal{G}(-d) \rightarrow \mathcal{G}$  induces the exact sequence  $0 \rightarrow \mathcal{G}(-d) \rightarrow \mathcal{G} \rightarrow G \rightarrow 0$ .*

*Proof.* Since  $Y$  is ACM,  $H^1(I_{Y/X}(-d + \nu)) = 0, \forall \nu$ , hence in the sequence stated in the previous lemma, the right hand map is surjective on the level of sections. Therefore, the map  $\oplus_{i=1}^k \mathcal{O}_X(-m_i) \rightarrow I_{Y/X}$  can be lifted to a map  $\oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i) \rightarrow I_{Y_2/X_2}$ . Since a global section of  $I_{Y_2/X_2}(\nu)$  maps to zero in  $I_{Y/X}$  only if it is a multiple of  $f$ , by Nakayama's lemma, this lift is surjective at the level of global sections in different twists, and hence on the level of sheaves. Hence there is a commuting diagram of exact sequences:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ I_{Y_2/X_2}(-d) & \rightarrow & I_{Y_2/X_2} & \rightarrow & I_{Y/X} & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ \oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i - d) & \rightarrow & \oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i) & \rightarrow & \oplus_{i=1}^k \mathcal{O}_X(-m_i) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ \mathcal{G}(-d) & \rightarrow & \mathcal{G} & \rightarrow & G & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

where the sheaf  $\mathcal{G}$  is defined as the kernel of the lift, and the map from the left column to the middle column is multiplication by  $f$ . It is easy to verify that the lowest row induces an exact sequence

$$0 \rightarrow G(-d) \rightarrow \mathcal{G} \rightarrow G \rightarrow 0.$$

By Nakayama's lemma,  $\mathcal{G}$  is a vector bundle on  $X_2$ . □

**Proposition 2.** *Let  $E$  be an ACM bundle on  $X$ . If  $E$  extends to a bundle  $\mathcal{E}$  on  $X_2$ , then  $E$  is a sum of line bundles.*

*Proof.* There is an exact sequence  $0 \rightarrow E(-d) \rightarrow \mathcal{E} \rightarrow E \rightarrow 0$ , where the left hand map is induced by multiplication by  $f$  on  $\mathcal{E}$ . Let  $F_0 = \oplus \mathcal{O}_{\mathbb{P}^n}(a_i) \twoheadrightarrow E$  be a surjection induced by the minimal generators of  $E$ . Since  $E$  is ACM, this lifts to a map  $F_0 \twoheadrightarrow \mathcal{E}$ . This lift is surjective on global sections by Nakayama's lemma (since the sections of  $\mathcal{E}$  which are sent to 0 in  $E$  are multiples of  $f$ ). Thus we have a diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & E(-d) & & \\
 & & & & \downarrow & & \\
 0 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & \mathcal{E} \rightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & G_1 & \rightarrow & F_0 & \rightarrow & E \rightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & E(-d) & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

$G_1$  and  $F_1$  are sums of line bundles on  $\mathbb{P}^n$  by Horrocks' Theorem. Furthermore,  $G_1 \cong F_0(-d)$ . Thus  $0 \rightarrow F_0(-d) \xrightarrow{\Phi} F_0 \rightarrow E \rightarrow 0$  is a minimal resolution for  $E$  on  $\mathbb{P}^n$ . As a consequence of this, one checks that  $\det \Phi = f^{\text{rank } E}$ . On the other hand, the degree of  $\det \Phi = d \text{rank } F_0$  and so we have  $\text{rank } F_0 = \text{rank } E$ . Restricting, this resolution to  $X$ , we get a surjection  $F_0 \otimes \mathcal{O}_X \rightarrow E$ . The ranks of both vector bundles being the same, this implies that this is an isomorphism. □

**Corollary 1.** *Let  $Y \subset X$  be a codimension 2 ACM subvariety. If the conormal sheaf sequence (1) splits, then*

- the ACM bundle  $G$  associated to  $Y$  is a sum of line bundles,
- there is a codimension 2 subvariety  $S$  in  $\mathbb{P}^n$  such that  $Y = X \cap S$ .

*Proof.* The first statement follows from Lemma 2 and Proposition 2. For the second statement, since the bundle  $G$  associated to  $Y$  is a sum of line bundles  $\oplus_{i=1}^{k-1} \mathcal{O}_X(-l_i)$  on  $X$ , the map  $G \rightarrow \oplus_{i=1}^k \mathcal{O}_X(-m_i)$  can be lifted to a map  $\oplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}}(-l_i) \rightarrow \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}}(-m_i)$ . The determinantal variety  $S$  of codimension 2 in  $\mathbb{P}^n$  determined by this map has the property that  $Y = X \cap S$ . □

In conclusion, we obtain the following collection of counterexamples:

**Corollary 2.** *If  $G$  is an ACM bundle on  $X$  which is not a sum of line bundles, and if  $Y$  is a subvariety of codimension 2 in  $X$  constructed from  $G$  as in Proposition 1, then  $Y$  does not satisfy the conclusion of either Question 2 or Question 3.*

Buchweitz-Greuel-Schreyer have shown [BGS] that any hypersurface of degree at least 2 supports (usually many) non-split ACM bundles. We will give another construction in the next section.

## 3. REMARKS

3.1. The infinitesimal Question 3 was treated by studying the extension of the bundle to the thickened hypersurface  $X_2$ . This method goes back to Ellingsrud, Gruson, Peskine and Strømme [EGPS]. If we are not interested in the infinitesimal Question 3, but just in the more geometric Question 2, a geometric argument gives an even easier proof of the existence of codimension 2 ACM subvarieties  $Y \subset X$  which are not of the form  $Y = X \cap Z$  for some codimension 2 subvariety  $Z \subset \mathbb{P}^n$ .

**Proposition 3.** *Let  $E$  be an ACM bundle on a hypersurface  $X$  in  $\mathbb{P}^n$  which extends to a sheaf  $\mathcal{E}$  on  $\mathbb{P}^n$ ; i.e. there is an exact sequence*

$$(2) \quad 0 \rightarrow \mathcal{E}(-d) \xrightarrow{f} \mathcal{E} \rightarrow E \rightarrow 0$$

*Then  $E$  is a sum of line bundles.*

*Proof.* At each point  $p$  on  $X$ , over the local ring  $\mathcal{O}_{\mathbb{P},p}$  the sheaf  $\mathcal{E}$  is free, of the same rank as  $E$ . Hence  $\mathcal{E}$  is locally free except at finitely many points. Let  $\mathbb{H}$  be a general hyperplane not passing through these points. Let  $X' = X \cap \mathbb{H}$ , and  $\mathcal{E}', E'$  be the restrictions of  $\mathcal{E}, E$  to  $\mathbb{H}, X'$ .

It is enough to show that  $E'$  is a sum of line bundles on  $X'$ . This is because any isomorphism  $\oplus \mathcal{O}_{X'}(a_i) \rightarrow E'$  can be lifted to an isomorphism  $\oplus \mathcal{O}_X(a_i) \rightarrow E$ , as  $H^1(E(\nu)) = 0, \forall \nu \in \mathbb{Z}$ . The bundle  $E'$  on  $X'$  is ACM and from the sequence

$$0 \rightarrow \mathcal{E}'(-d) \rightarrow \mathcal{E}' \rightarrow E' \rightarrow 0,$$

it is easy to check that  $H^i(\mathcal{E}'(\nu)) = 0, \forall \nu \in \mathbb{Z}$ , for  $2 \leq i \leq n-2$ . Since  $\mathcal{E}'$  is a vector bundle on  $\mathbb{H}$ , we can dualize the sequence to get

$$0 \rightarrow \mathcal{E}'^\vee(-d) \rightarrow \mathcal{E}'^\vee \rightarrow E'^\vee \rightarrow 0.$$

$E'^\vee$  is still an ACM bundle, hence  $H^i(\mathcal{E}'^\vee(\nu)) = 0, \forall \nu \in \mathbb{Z}$ , and  $2 \leq i \leq n-2$ .

By Serre duality, we conclude that  $\mathcal{E}'$  is an ACM bundle on  $\mathbb{H}$ , and by Horrocks' theorem,  $\mathcal{E}'$  is a sum of line bundles. Hence, its restriction  $E'$  is also a sum of line bundles on  $X'$ .  $\square$

**Proposition 4.** *Let  $Y$  be an ACM subvariety of codimension 2 in the hypersurface  $X$  such that the associated ACM bundle  $G$  is not a sum of line bundles. Then there is no pure subvariety  $Z$  of codimension 2 in  $\mathbb{P}^n$  such that  $Z \cap X = Y$ .*

*Proof.* Suppose there is such a  $Z$ . Then there is an exact sequence  $0 \rightarrow I_{Z/\mathbb{P}}(-d) \rightarrow I_{Z/\mathbb{P}} \rightarrow I_{Y/X} \rightarrow 0$ , where the inclusion is multiplication by  $f$ , the polynomial defining  $X$ . Since  $Z$  has no embedded points,  $H^1(I_{Z/\mathbb{P}}(\nu)) = 0$  for  $\nu \ll 0$ . Combining this with  $H^1(I_{Y/X}(\nu)) = 0, \forall \nu \in \mathbb{Z}$ , and using the long exact sequence of cohomology, we get  $H^1(I_{Z/\mathbb{P}}(\nu)) = 0, \forall \nu \in \mathbb{Z}$ .

Now suppose  $Y$  has the resolution  $0 \rightarrow G \rightarrow \oplus \mathcal{O}_X(-m_i) \rightarrow I_{Y/X} \rightarrow 0$ . From the vanishing just proved, the right hand map can be lifted to a map  $\oplus \mathcal{O}_{\mathbb{P}}(-m_i) \rightarrow I_{Z/\mathbb{P}}$ , which is easily checked to be surjective (at the level of global sections). It follows that if  $\mathcal{G}$  is the kernel of this lift,  $\mathcal{G}$  is an extension of  $G$  to  $\mathbb{P}^n$ . By the previous proposition,  $G$  is a sum of line bundles. This is a contradiction.  $\square$

3.2. Voisin's original example was as follows. Let  $P_1$  and  $P_2$  be two planes meeting at a point  $p$  in  $\mathbb{P}^4$ . The union  $\Sigma$  is a surface which is not locally Cohen-Macaulay at  $p$ . Let  $X$  be a smooth hypersurface of degree  $d > 1$  which passes through  $p$ .  $X \cap \Sigma$  is a curve  $Z$  in  $X$  with an embedded point at  $p$ . The reduced subscheme  $Y$  has the form  $Y = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are plane curves. Voisin argues that  $Y$  itself does not have the form  $X \cap S$  for any surface  $S$  in  $\mathbb{P}^4$ .

We can treat this example from the point of view of ACM bundles.  $I_{Z/X}$  has a resolution on  $X$  which is just the restriction of the resolution of the ideal of the union  $P_1 \cup P_2$  in  $\mathbb{P}^4$ , viz.

$$0 \rightarrow \mathcal{O}_X(-4) \rightarrow 4\mathcal{O}_X(-3) \rightarrow 4\mathcal{O}_X(-2) \rightarrow I_{Z/X} \rightarrow 0.$$

From the sequence  $0 \rightarrow I_{Z/X} \rightarrow I_{Y/X} \rightarrow k_p \rightarrow 0$ , it is easy to see that  $Y$  is ACM, with a resolution

$$0 \rightarrow G \rightarrow 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d) \rightarrow I_{Y/X} \rightarrow 0.$$

$G$  is an ACM bundle. If it were a sum of line bundles, comparing the two resolutions, we find that  $h^0(G(2)) = 0$  and  $h^0(G(3)) = 4$ , hence  $G = 4\mathcal{O}_X(-3)$ . But then  $G \rightarrow 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d)$  cannot be an inclusion. Thus  $G$  is an ACM bundle which is not a sum of line bundles.

Voisin's subsequent smooth examples were obtained by placing  $Y$  on a smooth surface  $T$  contained in  $X$  and choosing divisors  $Y'$  in the linear series  $|Y + mH|$  on  $T$ . When  $m$  is large,  $Y'$  can be chosen smooth. In fact, such curves  $Y'$  are doubly linked to the original curve  $Y$  in  $X$ , hence they have a similar resolution  $G' \rightarrow L \rightarrow I_{D'/X} \rightarrow 0$ , where  $L$  is a sum of line bundles and where  $G'$  equals  $G$  up to a twist and a sum of line bundles.

The fact that  $G$  above is not a sum of line bundles is related (via the mapping cone of the map of resolutions) to the fact that  $k_p$  itself cannot have a finite resolution by sums of line bundles on  $X$ . This follows from the following proposition which provides another argument for the existence of ACM bundles on arbitrary smooth hypersurfaces of degree  $\geq 2$ .

**Proposition 5.** *Let  $X$  be a smooth hypersurface in  $\mathbb{P}^n$  of degree  $\geq 2$  with homogeneous co-ordinated ring  $S_X$ . Let  $L$  be a linear space (possibly a point or even empty) inside  $X$  of codimension  $r$ , with homogeneous ideal  $I(L)$  in  $S_X$ . A free presentation of  $I(L)$  of length  $r - 2$  will have a kernel whose sheafification is an ACM bundle on  $X$  which is not a sum of line bundles.*

*Proof.* It should first be understood that the homogeneous ideal  $I(L)$  of the empty linear space will be taken as the irrelevant ideal  $(X_0, X_1, \dots, X_n)$ . Let the free presentation of  $I(L)$  together with the kernel be

$$0 \rightarrow M \rightarrow F_{r-2} \rightarrow \cdots \rightarrow F_0 \rightarrow I(L) \rightarrow 0,$$

where  $F_i$  are free graded  $S_X$  modules. Its sheafification looks like

$$0 \rightarrow \tilde{M} \rightarrow \tilde{F}_{r-2} \rightarrow \cdots \rightarrow \tilde{F}_0 \rightarrow I_{L/X} \rightarrow 0.$$

Since  $L$  is locally Cohen-Macaulay,  $\tilde{M}$  is a vector bundle on  $X$ , and since  $L$  is ACM, so is  $\tilde{M}$ .  $M$  equals  $\bigoplus_{\nu \in \mathbb{Z}} H^0(\tilde{M}(\nu))$ . Hence,  $\tilde{M}$  is a sum of line bundles only if  $M$  is a free  $S_X$  module.

If  $\mathbb{H}$  is a general hyperplane in  $\mathbb{P}^n$  which meets  $X$  and  $L$  transversally along  $X_{\mathbb{H}}$  and  $L_{\mathbb{H}}$  respectively, the above sequences of modules and sheaves can be restricted to give similar sequences in  $\mathbb{H}$ . The restriction  $\tilde{M}_{\mathbb{H}}$  is an ACM bundle on  $X_{\mathbb{H}}$ .

Repeat this successively to find a maximal and general linear space  $\mathbb{P}$  in  $\mathbb{P}^n$  which does not meet  $L$ . If  $X' = X \cap \mathbb{P}$ , the restriction of the sequence of  $S_X$  modules to  $X'$  gives a resolution

$$0 \rightarrow M' \rightarrow F'_{r-2} \rightarrow \cdots \rightarrow F'_0 \rightarrow S_{X'} \rightarrow k \rightarrow 0.$$

Localize this sequence of graded  $S_{X'}$  modules at the irrelevant ideal  $I(L) \cdot S_{X'}$ , to look at its behaviour at the vertex of the affine cone over  $X'$ .  $k$  is the residue field of this local ring. Since  $X$  and hence  $X'$  has degree  $\geq 2$ , the cone is not smooth at the vertex. By Serre's theorem ([Se], IV-C-3-Cor 2),  $k$  cannot have finite projective dimension over this local ring. Hence  $M'$  is not a free module. Therefore neither is  $M$ .  $\square$

3.3. We make a few concluding remarks about Question 1, the Degree Conjecture of Griffiths and Harris. A vector bundle  $G$  on a smooth hypersurface  $X$  in  $\mathbb{P}^4$  has a second Chern class  $c_2(G) \in A^2(X)$ , the Chow group of codimension 2 cycles. If  $h \in A^1(X)$  is the class of the hyperplane section of  $X$ , the degree of any element  $c \in A^2(X)$  will be defined to be the degree of the zero cycle  $c \cdot h \in A^3(X)$ . (Note that by the Lefschetz theorem, all classes in  $A^1(X)$  are multiples of  $h$ .)

With this notation, if  $E$  is any bundle on  $X$  and  $Y$  is a curve obtained from  $E$  with the sequence (*vide* Proposition 1)

$$0 \rightarrow E \rightarrow \oplus_{i=1}^k \mathcal{O}_X(m_i) \rightarrow I_{Y/X}(m) \rightarrow 0,$$

a calculation tells us that the degree  $d$  of  $X$  divides the degree of  $Y$  if and only if  $d$  divides the degree of  $c_2(E)$ .

More generally: let  $Y$  be any curve in  $X$  and resolve  $I_{Y/X}$  to get

$$0 \rightarrow E \rightarrow \oplus_{i=1}^l \mathcal{O}_X(b_i) \rightarrow \oplus_{i=1}^k \mathcal{O}_X(a_i) \rightarrow I_{Y/X} \rightarrow 0,$$

where  $E$  is an ACM bundle on  $X$ . Then a similar calculation tells us that the degree  $d$  of  $X$  divides the degree of  $Y$  if and only if  $d$  divides the degree of  $c_2(E)$ .

Hence we may ask the following question which is equivalent to the Degree Conjecture:

**ACM Degree Conjecture.** *If  $X$  is a general hypersurface in  $\mathbb{P}^4$  of degree  $d \geq 6$ , then for any indecomposable ACM vector bundle  $E$  on  $X$ ,  $d$  divides the degree of  $c_2(E)$ .*

The examples created above in Proposition 5 satisfy this, when  $L$  has codimension  $> 2$  in  $X$ . In [MRR], this conjecture is settled for ACM bundles of rank 2 on  $X$ .

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